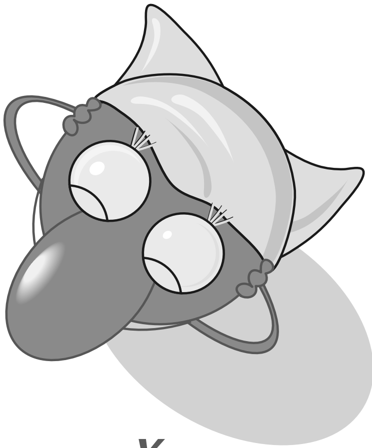
Math Challenge #45 THREE SQUARES

Figure This! Math Challenges for Families

When do two squares make a new square?



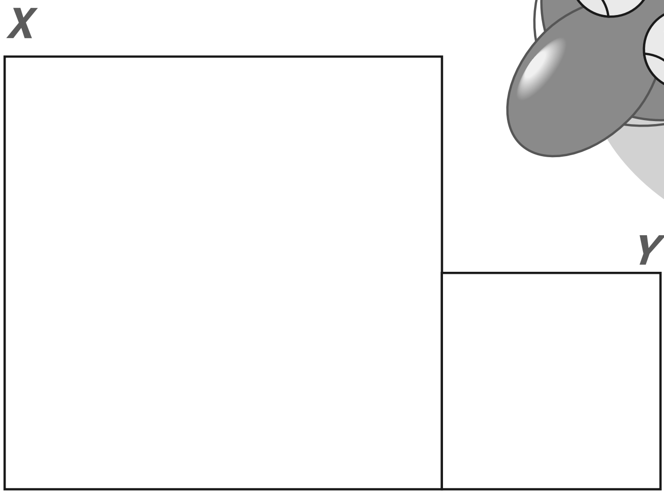


Figure This! Can you make a new square from two squares?

Hint: Cut two squares from a sheet of paper and tape them together as in the diagram. Find a point *Z* along the bottom of the two squares so that angle *XZY* is a right angle. Then use a pair of scissors.

The Pythagorean Theorem states that for any right triangle, $a^2 + b^2 = c^2$, where c is the length of the longest side, and a and b are the lengths of the other two sides. This relationship is often used to find the distance between two points and is fundamental in construction, engineering, and the sciences.

Answer:
Draw segments *XZ* and *YZ* as in the picture. The three pieces can be formed into a square as shown. Any two squares can be used to make a third square in this manner.

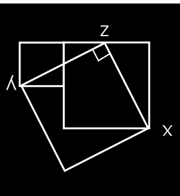


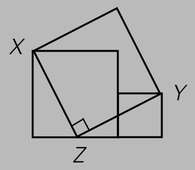
Figure This!

Get Started:

To locate point Z as described in the hint, place another sheet of paper on top of the two squares. Place one corner of the sheet on the bottom of the two squares. Slide the paper until one edge touches point X and an adjacent edge touches point Y. Cut the triangle pieces off and try to form a square by re-arranging all pieces.

Complete Solution:

The Hint and the Answer suggest using a piece of paper to find point Z and construct a square. (The edges of the paper will lie on two sides of the new square. Carpenters would use a carpenter's square for finding this point.)



A different way to find point Z is to mark off the length of a side of the small square along the bottom of the larger square starting at the left. This length locates point Z. Because the total length of the bottom is the sum of the lengths of a side of each square, Z also separates the bottom into two lengths that are the lengths of the sides of the squares. Draw segments XZ and YZ. The two right triangles are the same size and shape because each has a right angle and the two smaller sides are each a length of the original squares. Because the triangles are the same size and shape, XZ and YZ are the same length. They become sides of the new square. Using the angles of the triangles along the base, angle XZY can be shown to be a right angle.

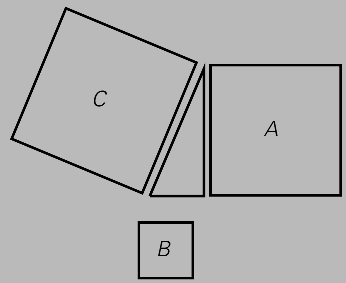
Try This:

- Make a square with 3-inch sides and another square with 4-inch sides. Use these two squares to make a new square. How long is each side of the new square?
- Show that segment XY in the Challenge is the diameter of a circle that passes through point Z.

Additional Challenges:

(Answers located in back of booklet)

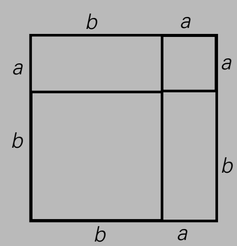
1. A Pythagorean triple is a set of three counting numbers a, b, and c so that $a^2 + b^2 = c^2$. For example, the numbers 3, 4, and 5 make a Pythagorean triple because $3^2 + 4^2 = 5^2$, or $9 + 16 = 25$. Do the numbers 6, 8, and 10 make a Pythagorean triple?
2. Find a different Pythagorean triple that contains 5.
3. Find a Pythagorean triple in which 7 is the smallest number and the larger numbers differ by 1.
4. Can all of the numbers in a Pythagorean triple be odd?
5. This diagram shows a right triangle and three squares, A, B, and C. The sides of each square have the same length as one of the sides of the triangle.



Make a larger square using square C and four copies of the right triangle. Make another large square from squares A and B and four copies of that same right triangle. Use the new square to show that the sum of the areas of squares A and B equals the area of square C.

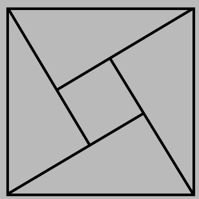
Things to Think About:

- Ancient Egyptians were able to create right angles for building and surveying using a rope with 12 equally-spaced knots to form a right triangle.
- Any triangle can be cut and reformed into a rectangle.
- The two smaller numbers of a Pythagorean triple cannot both be odd.
- Any multiple of a Pythagorean triple is also a Pythagorean triple.
- The Pythagorean Theorem holds for any numbers a, b, and c that form the sides of a right triangle.
- If two right triangles have corresponding sides the same length, the triangles have to be the same size and shape (congruent).
- This diagram shows that $(a + b)^2 = a^2 + 2ab + b^2$



Did You Know That?

- Pythagoras was a Greek mathematician (about 580–500 BC) who formed a secret society based partly on mathematical discoveries. The motto of his followers is said to have been "All is number."
- What is called the Pythagorean Theorem was known to the Egyptians as early as 2000 BC and to the Babylonians in 1700 BC.
- The Hindu mathematician Bhaskara (about 1114-1185) proved the Pythagorean Theorem simply by drawing this picture and saying "Behold!"



- Lewis Carroll, author of *Alice in Wonderland*, and US President James Garfield wrote proofs of the Pythagorean Theorem.
- French mathematician Pierre Fermat (1601–1665) built on the Pythagorean Theorem by proving that no cube is the sum of two cubes and no fourth power is the sum of two fourth powers. He believed that the same was true for all whole-number powers greater than 2. He wrote in the margin of a notebook that he had a proof of his belief, but the margin was too small for the proof. Only in 1995, more than 300 years later, did British-born mathematician Andrew Wiles find the key that proved Fermat correct.
- Choose any two different positive counting numbers a and b with a the bigger of the two. If $x = 2ab$, $y = a^2 - b^2$ and $z = a^2 + b^2$, then x, y, and z form a Pythagorean triple.

Resources:
Books:

- Bell, E. *The Last Problem*. Washington, DC: Mathematical Association of America, 1990.
- Loomis, E. *The Pythagorean Proposition*. Reston, VA: The National Council of Teachers of Mathematics, 1940.

Websites:

- **cut-the-knot.com/pythagoras/index.html**
- **www.geom.umn.edu/-demo5337**

Notes:

Axis

Math Challenge

#46

TIME ZONES

Figure This!

Math Challenges for Families

It's afternoon in San Francisco, what time is it in Cairo, Egypt?

Standard Time Zones of the World

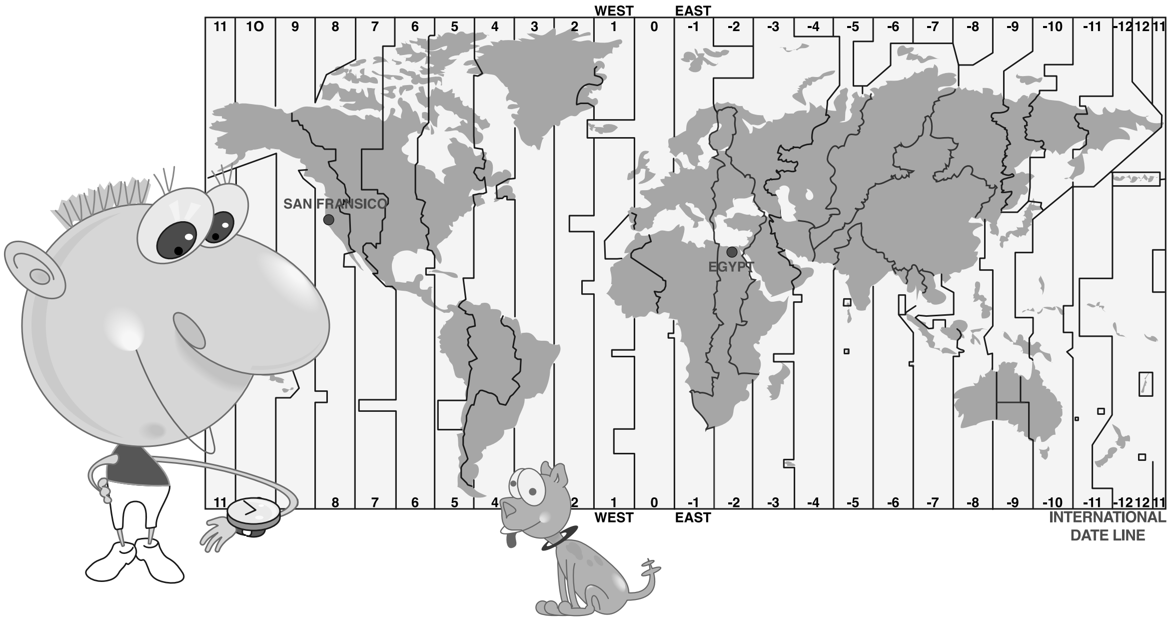


Figure This! After Helix gets home from school in San Francisco, can he call his grandfather in Cairo, Egypt, and expect to find him awake?

Hint: Think about the number of time zones separating San Francisco and Cairo, Egypt.

Determining the time in different regions of the world involves the use of positive and negative values. A similar approach is sometimes used to describe profit and loss, or to describe changes in climate.

Answer:

There is a 10-hour time difference between San Francisco and Cairo, Egypt. If Helix called at 3:00 PM from California, it would be 1:00 AM the next day in Egypt. His grandfather might be asleep.

Figure This!

Get Started:

Find San Francisco, California and Cairo, Egypt on a world map with time zones marked. How many time zones separate the two cities? When school ends in San Francisco, what time is it in Cairo?

Complete Solution:

- Global time is calculated from an imaginary line, called the Prime Meridian that extends from the North Pole to the South Pole through Greenwich, England. Each time zone to the east of Greenwich is numbered with a negative (-) number; while each zone to the west is numbered with a positive (+) number. Cairo is at –2, which indicates it is two time zones east of Greenwich. San Francisco is at +8, which indicates it is 8 time zones west of Greenwich. This means that a total of 10 time zones separates the two cities. Therefore, 3:00 PM in San Francisco is 1:00 AM the next morning in Cairo.
- Another way to do this challenge follows: San Francisco is at +8 on the map. Cairo is at –2. Using mathematical operations, $+ 8 - (-2) = 10$. When it is 3:00 PM in San Francisco, it is 10 hours later in Cairo, or 1:00 AM the next morning.

Try This:

- Using a marking pen, draw the points on a basketball or a beachball that represent the North Pole, the South Pole, San Francisco, and Cairo. Locate Greenwich, England and draw a line from the North Pole to the South Pole through Greenwich. To represent each time zone, draw more lines from the North Pole to the South Pole at intervals of 15˚, until you reach Greenwich again. Count the number of time zones between San Francisco and Cairo.
- Look in a telephone directory for a chart of time zones. Use the chart to determine what the current time is in each time zone of the United States.

Additional Challenges:

(Answers located in back of booklet)

1. What would be a reasonable time to call Cairo from San Francisco?
2. The International Date Line is located halfway around the world from Greenwich, England, at about the 180th meridian. When crossing from the west, the date is advanced one day. When crossing from the east, the date is set back one day. If it is noon on January 1 in San Francisco, what is the date and time in Singapore?
3. The New York Stock Exchange is open from 9:30 AM to 4:00 PM Eastern Standard Time. What are the hours in the Pacific Time Zone when the New York Stock Exchange is open?
4. Are there places in the world where the difference in time zones is 20 hours or more?

Things to Think About:

- What does the earth's rotation have to do with time zones?
- Why is there an International Date Line?
- Why were the time zones established by using 15˚ angles?
- Why is there Daylight Saving Time?
- It is possible to leave one city and arrive in another city "before" you left the original city.

Did You Know That?

- The United States covers six time zones: Eastern, Central, Mountain, Pacific, Alaska, and Hawaii-Aleutian. US territories cover four more time zones. Russia covers 11 time zones.
- In order to accommodate local geography, the boundaries indicating actual time zones are not straight lines.
- Some US states, such as Indiana, are in two time zones.
- Two US territories, Guam and Wake Island, are on the other side of the International Date Line from North America.
- In the US, Daylight Saving Time, which first originated during World War I, begins at 2:00 AM on the first Sunday in April and ends at 2:00 AM on the last Sunday in October.
- Hawaii and Arizona do not observe Daylight Saving Time.
- A 24-hour clock is used by the military and most scientists. In the 24-hour time system, the hours are numbered from 0 to 23, and there are no AM and PM designations. With this time system, 0 o'clock is possible.
- A typical time zone differs from its neighboring zones by one hour, but some time zones differ by a fraction of an hour. For example, the time on the Canadian island of St. John's differs from that in the rest of Newfoundland by 30 minutes. The time in Nepal differs from neighboring India by 15 minutes.

Resources:

Books:

- *The World Almanac and Book of Facts 2000*. Mahwah, NJ: World Almanac Books, 1999.

Websites:

- **www.askjeeves.com.**
- **www.cstv.to.cnr.it/toi/uk/timezone.html**
- **www.lib.utexas.edu/Libs/PCL/Map_collection/world_maps/World_Time_ref802649_1999.pdf**
- **time.greenwich2000.com**

Math Challenge

#47

VCR

FigureThis!
Math Challenges for Families

Do you always get 6 hours of recording on a 6-hour tape?

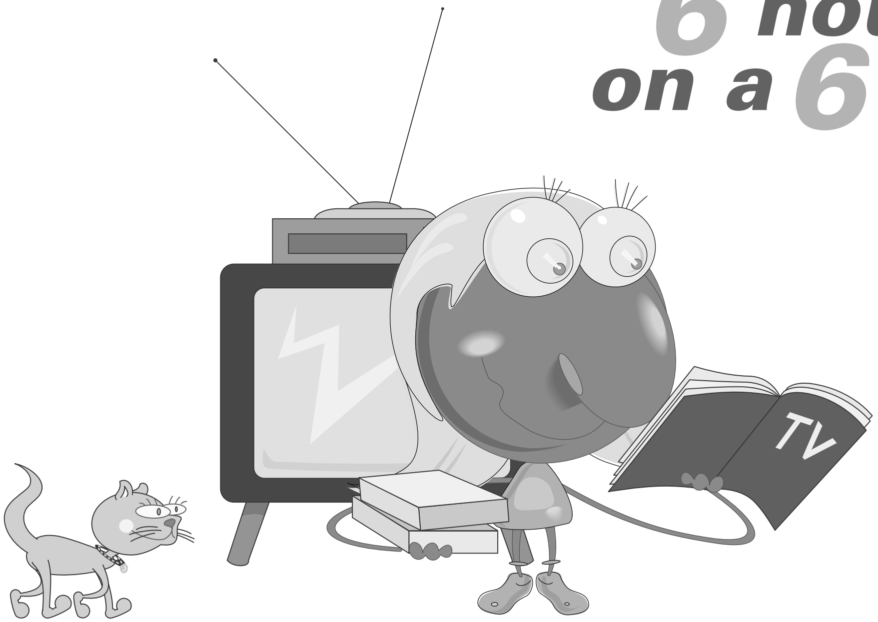


Figure This! Suppose the setting SP (standard play) on a VCR allows 2 hours of recording with an ordinary 120-minute tape. Changing the setting to EP (extended play) allows 6 hours of recording. After taping a 30-minute show on SP, the VCR is reset to EP. How many more 30-minute shows can be recorded on this tape?

Hint: What fraction of the tape was used to record the first show?

Proportional reasoning involves simple fractions, ratios, and probabilities. It is used by people comparing grocery prices, economists describing population densities, architects making scale models, and chemists determining the relative strengths of solutions.

Answer:
Nine more shows.

Figure This!

Get Started:

How many half-hour shows could be recorded on the SP setting? What part of the tape was left after the first half-hour show was recorded?

Complete Solution:

- You can consider this problem in terms of minutes or in units of half-hour shows. Using only the SP setting, the tape can record 120 minutes of shows. If one 30 minute show has already been recorded, 1/4 of the tape has been used.

Type of Setting	Minutes Used	Total Minutes Available on Tape	Part of Tape Used
SP	30	120	30/120, or 1/4
EP	(1/4) • 360, or 90	360	90/360 or 1/4

With 1/4 of the tape filled, 90 minutes of EP has been used, so that there are 360 – 90 or 270 min left. At 30 minutes per show, nine additional shows can be taped.

- If you are using half-hour units, there are four half–hour shows on SP in two hours. One half–hour show used 1/4 of the tape. Three-fourths of the tape is left. With EP, you can do 12 half–hour shows, and then you have enough tape for 3/4 • 12 or 9 shows remaining.
- Yet another way to consider this problem is to use proportions as follows:

$$\frac{\text{30 minutes}}{\text{120 minutes available on SP}} = \frac{\text{? minutes}}{\text{360 minutes available on EP}}$$

Because 3 • 120 = 360 and 3 • 30 = 90, there are 90 minutes used on the EP setting, so that there are 360 – 90, or 270 minutes available which allows for 9 thirty-minute shows on EP.

Similarly 30 • 360 = 120 • (number of minutes) so the number of minutes is 90.

Try This:

- Select a favorite half-hour TV show. Set your VCR to EP, then record only the program itself, editing out all the commercials and stopping the VCR as soon as the show ends. How much time is used for commercials? How many programs could you get on a 6 hour tape without commercials and credits?
- Select two different channels. Time the news and the amount of time for commercials on the channels. How do the proportions of commercials and news differ?
- Think about a movie that you have seen both in a theater and on television. Do the movies in the different settings take the same amount of time? How do the times compare with the same movie on a rental tape?
- What proportion of Saturday morning TV shows are cartoons?

Additional Challenges:

(Answers located in back of booklet)

1. Some VCRs have a third setting, LP, which allows 4 hours of recording on an ordinary 120-minute tape. Suppose you taped one 30-minute show on SP, changed the setting to LP, then taped another 30-minute show. If you then change the setting to EP, how many more 30-minute shows would fit on the tape?

2. The table shows average TV watching times, in hours and minutes, for women, men, and teenagers in May 1998 based on a random sample of the population. Of the time that each group spends watching TV (on average), which group spends the largest proportion on Saturday morning?

Group	Mon-Fri 10:00 AM-4:30 PM	Mon-Fri 4:30 PM-7:30 PM	Mon-Sun 8:00 PM-11:00 PM	Sat 7:00 AM-1:00 PM	Mon-Fri 11:30 PM-1:00 AM
Women (18 and older)	5 hr. 41 min.	3 hr. 47 min.	9 hr. 1 min.	42 min.	1 hr. 32 min.
Men (18 and older)	3 hr. 19 min.	2 hr. 47 min.	8 hr. 11 min.	35 min.	1 hr. 26 min.
Teenagers (12-17)	1 hr. 59 min.	2 hr. 56 min.	5 hr. 51 min.	42 min.	50 min.

3. Use the chart in Number 2. Of the time that each group spends watching TV (on average), which group spends the largest proportion watching primetime (Monday - Sunday from 8 PM to 11 PM)?

Things to Think About:

- Proportions can be helpful for determining the heights of objects that cannot be easily measured. For example, the approximate height of a flagpole, tree, or building can be determined using shadows and a known measure, such as your height.
- In 1997, 98% of US households owned at least one television set. This represented about 98 million homes. Of these, 84% also owned a VCR.

Did You Know That?

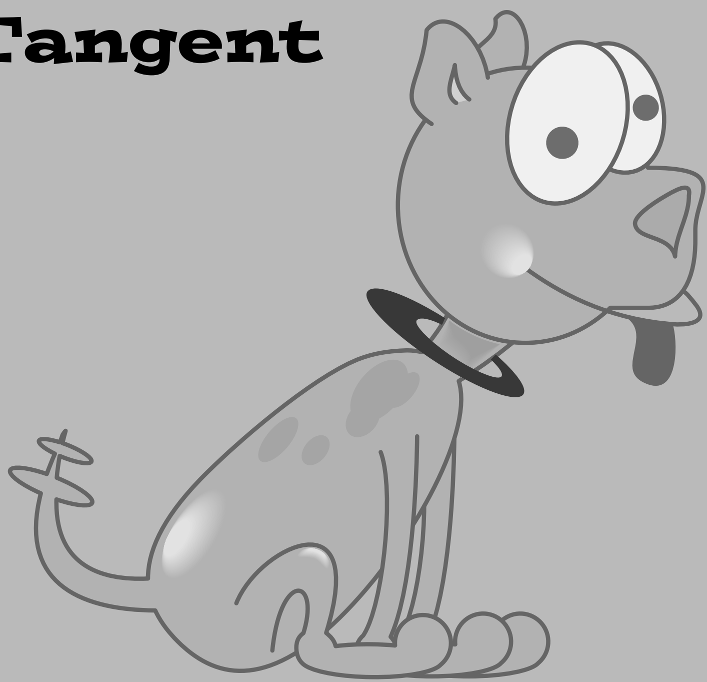
- Videocassettes are available in several different lengths, including one that can record up to 8 hours on EP.
- Videodiscs are less expensive to produce than cassettes, and offer superior sound, color, and picture quality.
- A camcorder combines a VCR and a video camera in one machine.

Resources:
Books:

- *Encyclopedia Americana, Vol. 28*, Grolier Inc., Danbury, Connecticut, 1998.
- *Collier's Encyclopedia, Vol. 23*, New York: Nuffield, Publications, 1997.
- *The World Almanac and Book of Facts 2000*. Mahwah, NJ: World Almanac Books, 1999.

Notes:

Tangent



Math Challenge

#48

WORKING HOURS

Figure This! Math Challenges for Families

How much time do teens spend on the job?

Hours Worked by High School Seniors (per week)

Hours	Frequency
none	36%
1-5 hours	7%
6-10 hours	9%
11-15 hours	11%
16-20 hours	17%
21+ hours	21%

Source: National Center for Educational Statistics, National Assessment of Educational Progress. (Hours are rounded to the nearest whole number; percentages total more than 100 due to rounding.)

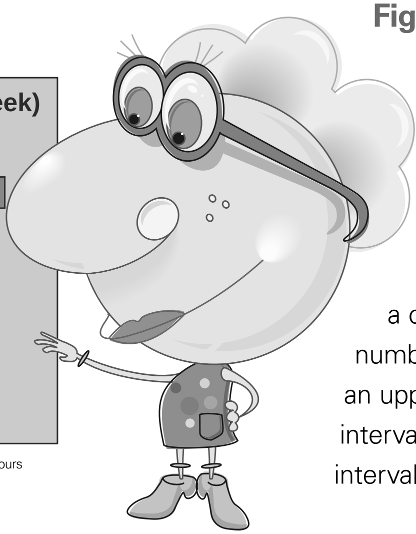


Figure This! What's the average number of hours high school seniors work per week?

Hint: Assume that there are 100 students in the survey data. The average (mean) is the total of all the hours a class of seniors worked divided by the total number of seniors. Statisticians usually assume an upper bound for each category that keeps the intervals the same width; in this case, the upper interval for the 21+ category could be 25.

An average is often used to summarize a set of numbers. Averages describe data on housing prices, wages, athletics, and academic performance.

Answer:

Between 9 and 11 hours per week depending on your approach to the problem.

Figure This!

Get Started:

If the data represent 100 students, how many students would fall in each category? How could you estimate the number of hours worked in each category? Making a chart may help you organize the information.

Complete Solution:

There are many ways to do this problem. The answer can only be approximated because of the way the data are reported.

- If you assume that the table shows data for 100 students, then the 7% who worked from one to five hours would correspond to seven students who worked one to five hours. To find an average (mean), you need to estimate the total number of hours worked. Consider the seven students who worked from one to five hours. Taken together, the least time they could have worked is seven hours. The greatest they could have worked is 35 hours. (The actual number is probably somewhere in between these values.) The first chart shows the least number of hours the 100 students could have worked, while the second chart shows the greatest number of hours they could have worked.

Least Number of Hours	Number of Students*	Least Number of Total Hours
0	36	0
1	7	7
6	9	54
11	11	121
16	17	272
21	21	441
Total	**100**	**895**

***Estimated**

Greatest Number of Hours	Number of Students*	Greatest Number of Total Hours
0	36	0
5	7	35
10	9	90
15	11	165
20	17	340
25*	21	525
Total	**100**	**1155**

***Estimated**

To find the lower value for the average number of hours worked per week, divide the total number of hours by the number of students: 895/100 = 8.95. The larger value for the average workweek can be found in the same way: 1155/100 = 11.55. Using this approach, an estimate for the average for a high school senior is between 8 1/2 hours and 12 1/2 hours.

- Another way to estimate the average uses an average for the hours worked in each category.

Smallest Number of Hours	Largest Number of Hours	Average Number of Hours	Number of Students*	Total Hours Worked
0	0	0	36	0
1	5	3	7	21
6	10	8	9	72
11	15	13	11	143
16	20	18	17	306
21	25*	23	21	483
		Total	**100**	**1025**

***Estimated**

To find a value for the average, divide the total number of hours by the number of students: 1025/100 = 10.25. This method results in an average of about 10 1/4 hours.

Try This:

- Survey some seniors in a high school near you to find how many hours per week they work. How do your results compare to the information in the challenge?
- Make up a set of numbers in which the median (or middle number) is the same as the mean or average.
- Make up a set of numbers in which the median is not the same as the mean or average.

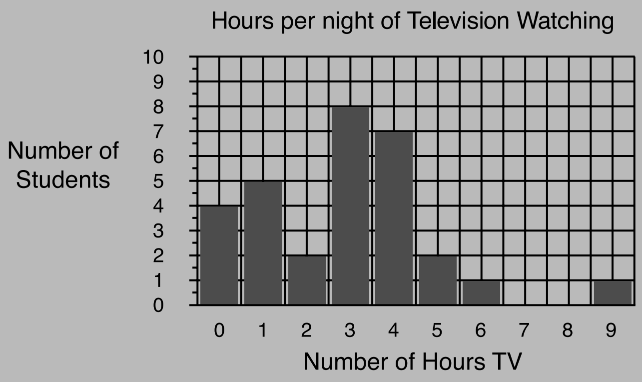
Additional Challenges:

(Answers located in back of booklet)

1. Would your estimate for the number of hours worked per week in the challenge change if you had used 200 students? 1000 students?
2. Which number should be omitted from the following set to result in an average of 20?

   12, 18, 8, 18, 21, 32, 25, 36, 25,10

3. This graph shows the results of a class survey of TV viewing habits. Use this data to find the average number of hours per night this class spent watching TV.



Things to Think About:

- Why is it important to use 0, representing those who said they did not work at all, in finding the average?
- Thirty-six percent of US high school seniors in the survey did not work at all, while 21% of them worked more than 21 hours per week.
- Is it possible for two very different sets of data to have the same mean?
- If there are one or more unusually large numbers with respect to the rest of the data in a data set, the mean may be much larger than seems typical. If there are one or more unusually small numbers in the set with respect to the rest of the data, the mean may be much smaller than seems typical.
- Why is it reasonable to use 25 as an upper bound for the 21+ category in the data of the challenge?

Did You Know That?

- In 1999, US teens watched television for an average of 12 1/4 hours per week (Nielson Media Research).
- The average earnings in 1998 for people 18 or older was $16,124 per year (without a high school diploma); $22,895 per year (with a high school diploma); and $40,478 per year (with a university degree). (US Census Bureau).
- On average, fast-food cooks make $6.29 an hour, service-station attendants make $7.34 an hour, machinists make $14.35 an hour, mail carriers make $16.39 an hour, and physical therapists make $27.49 an hour (US Bureau of Labor Statistics).
- In 1998, US production workers averaged 34.6 hours per week on the job. Their average hourly wage was $12.77; their average weekly earnings were $441.84 (US Bureau of Labor Statistics).
- In 1997, the US metropolitan area with the highest average annual salary was San Jose, California. Workers there earned an average of $48,702 per year (US Bureau of Labor Statistics).
- In 1998, men with a bachelor's degree earned an average of $50,272 per year, while women with a bachelor's degree earned $30,692 (US Census Bureau).
- On average, elephants in captivity live 40 years, guinea pigs 4 years, and opossums 1 year. Animals in the wild rarely live to their maximum potential life span.
- Whenever a set of data is summarized by an average or mean, a measure to indicate how the data are spread around the mean should be reported as well. One such measure is called the standard deviation. It describes a typical amount that the data values may differ from the mean.

Resources:

Books:

- *World Almanac and Book of Facts 2000*. Mahwah, NJ: World Almanac Books, 1999.
- *Current Population Reports.* Washington, DC: US Census Bureau, March 1998.

Websites:

National Center for Educational Statistics, National Assessment of Educational Progress:

- **nces.ed.gov/**

US Census Bureau:

- **www.census.gov/hhes/income/histine/p.16.html**

US Bureau of Labor Statistics:

- **stats.bls.gov/blshome.html**

Challenge 41:

1. 100,000.
2. 10,000.
3. Equally likely.

Challenge 42:

1. There are five different ways to score exactly 10 points, the same number of ways as for 11.
2. It is impossible to score 1 point. Any other number of points is possible.
3. 8, 11, 14, 17, 20, 22, 25, 28, 31, 36, 39, 42, 50, 53 or 64.
4. 28.

Challenge 43:

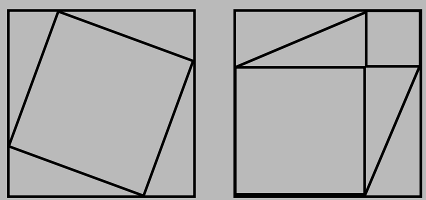
1. The stars could be in rows with 8, 9, 10, 11, and 12 stars.
2. 59
3. 301 students.

Challenge 44:

1. 16.
2. Arranging the tables in a 3 x 3 square leaves only 12 available seats.
3. Among the answers are 2 • 3 + 2(*n* - 2) and 2 (*n* + 1).

Challenge 45:

1. Yes, since $6^2 + 8^2 = 10^2$.
2. 5, 12, 13.
3. 7, 24, 25.
4. No.
5. The two large squares below have the same area.



Taking away the four right triangles from each large square shows that the remaining areas are equal.

Challenge 46:

1. Early in the morning or late at night. For example, 8:00 AM in San Francisco would be 6:00 PM in Cairo; 10:00 PM in San Francisco would be 8:00 AM in Cairo.
2. It is 4:00 AM on January 2.
3. 6:30 AM until 1:00 PM.
4. Yes. For example, the difference in time from Japan to Western Samoa is –20 hours.

Challenge 47:

1. 7 complete shows.
2. Teenagers (12 - 17).
3. Men (18 and older).

Challenge 48:

1. No. The estimated average would be the same in all three cases.
2. 25.
3. 2.87 hours per night.